4 Theory of Operators: I

1. We may recall what operators are from our earlier discussion:

$$\mathcal{O} |\eta\rangle = |\chi\rangle \tag{4.0.1}$$

2. Every observable quantity has an operator associated with it

3. Eigenvalues and Eigenvectors:

$$\mathcal{O} |\eta\rangle = \wr |\eta\rangle \tag{4.0.2}$$

   There are special vectors of this kind associated with every operator.

4. **Linear Operators:** Consider two operators $\hat{A}$ and $\hat{B}$. These operators are considered "linear operators" if:

   (a)
$$\left[\hat{A} + \hat{B}\right] |\eta\rangle = \hat{A} |\eta\rangle + \hat{B} |\eta\rangle \tag{4.0.3}$$

   (b) If c is a number (may be complex) then

$$\hat{A} [c |\eta\rangle] = c \left\{\hat{A} [|\eta\rangle]\right\} \tag{4.0.4}$$

   (c)
$$\hat{A} [|\eta\rangle + |\chi\rangle] = \hat{A} |\eta\rangle + \hat{A} |\chi\rangle \tag{4.0.5}$$

5. Note: The Hamiltonian operator is a linear operator. Why? Because the derivative operator is a linear operator.

6. The product of two operators $\hat{A}$ and $\hat{B}$ is defined as follows:

$$\hat{A}\hat{B} |\eta\rangle = \hat{A} \left[\hat{B} |\eta\rangle\right] = \hat{A} [|\chi\rangle] = |\alpha\rangle \tag{4.0.6}$$

   where we have assumed $\hat{B} |\eta\rangle \equiv |\chi\rangle$. Note this also defines the square of an operator:

$$\hat{A}^2 |\eta\rangle = \hat{A} \left[\hat{A} |\eta\rangle\right] \tag{4.0.7}$$

7. **Commutators:** The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{4.0.8}$$

8. Note that an operator $\hat{A}$ *commutes* with itself since

$$\left[\hat{A}, \hat{A}\right] = \hat{A}\hat{A} - \hat{A}\hat{A} = 0 \tag{4.0.9}$$

9. **Anti-Commutators:** The anti-commutator of two operators $\hat{A}$ and $\hat{B}$ is defined as

$$\left[\hat{A}, \hat{B}\right]_{+} = \hat{A}\hat{B} + \hat{B}\hat{A} \tag{4.0.10}$$

Note the "+"

10. **Homework:** Work out the following commutators:

    (a) $\left[\frac{d}{dx}, \frac{d}{dx}\right]$ = ?

    (b) $\left[x, \frac{d}{dx}\right]$= ?

    (c) $\left[f(x), \frac{d}{dx}\right]$ = ?

    (d) $\left[f(x), \frac{d}{dx}\right]_{+}$ = ?

    (e) $\left[\frac{d}{dx}, \frac{d}{dx}\right]_{+}$ = ?

    (f) $\left[\frac{d}{dx} + f(x)\right]^{2}$ = ?

11. **Eigenvalues and Eigenvectors**

$$\hat{A} \left|\eta\right\rangle = a \left|\eta\right\rangle \tag{4.0.11}$$

where a is a number, called the eigenvalue. $\left|\eta\right\rangle$ is the eigenvector.

12. **Representing operators**

    (a) Earlier we spoke about how we could "represent" vectors. That is, any vector can be represented as a linear combination of a complete set of vectors. (If this statement is not clear, please revise the linear algebra notes.)

    (b) Operators can be represented in a similar for. In fact if you have a complete set of vectors $\{\left|i\right\rangle\}$, we can write an operator as a matrix. What we mean by this is we could represent an operator using a collection of matrix elements that have the following form:

$$A_{j,l} \equiv \left\langle j \left| \hat{A} \right| l \right\rangle \tag{4.0.12}$$

    $A_{j,l}$ is the (j,l)-th element of the matrix that is used to represent the operator $\hat{A}$. (Make sure to compare this with the Pauli spin matrix homework so you understand whats going on clearly.)

    (c) Does this definition make sense? $\hat{A}\left|l\right\rangle$ is another vector. You could call it $\left|m\right\rangle$ if you like. In that case the right hand side of Eq. (4.0.12) is the "dot" product of two vectors: $\left\langle j\right|$ and $\left|m\right\rangle$. The "dot" product of two vectors is a number. Hence the definition in Eq. (4.0.12) makes sense. (If these arguments are not a 100% clear to you, you need to go back and revise handout on linear algebra and also the one dealing with representation theory.)

    (d) For Eq. (4.0.12) to be useful we should know what $\hat{A}$ does to $\left|l\right\rangle$ when it acts on it.

