

6 A Measurement is a Projection or a “dot” product (or inner product)!!

1. Lets go back and consider the Stern Gerlach experiment that we studied earlier:

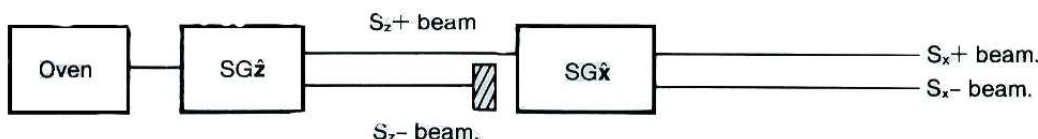


Figure 17:

2. After the second magnetic field, the molecules can only be seen to have an S_x^+ and S_x^- component.
3. As a homework convince yourself by running the java script on the course website that the probability of obtaining an $|SG_x^+\rangle$ or $|SG_x^-\rangle$ state as a result of the above experiment are both equal to $1/2$.
4. Now to understand this mathematically consider the following:

$$|SG_z^+\rangle \equiv \left\{ \sum_{i=1}^n |i\rangle \langle i| \right\} |SG_z^+\rangle = \sum_{i=1}^n c_i |i\rangle \quad (6.0.1)$$

We have of course seen this equation before. All we have done is “resolve the identity” in terms of the kets $\{|i\rangle\}$. But we could have equally well done the following:

$$|SG_z^+\rangle \equiv \{|SG_x^+\rangle \langle SG_x^+| + |SG_x^-\rangle \langle SG_x^-|\} |SG_z^+\rangle \quad (6.0.2)$$

Notice that the quantity inside $\{\dots\}$ is just equal to 1. (The resolution of the identity!!! See homework problem on page 37.)

5. How do we rationalize the above statement? We have noticed that the \hat{x}' and \hat{y}' vectors are “isomorphic” to the kets $|SG_x^+\rangle$ and $|SG_x^-\rangle$. And any vector in the x-y plane can be written as a linear combination of the \hat{x}' and \hat{y}' vectors. Hence these vectors form a *complete set* in “2-dimensions”. The resolution of the identity is simply a mathematical language that conveys the same meaning as the above set of words.
6. As a result of the above it must follow that $|SG_x^+\rangle$ and $|SG_x^-\rangle$ also form a complete set and hence the resolution of the identity in terms of these is valid.

7. We can simplify Eq. (6.0.2) by multiplying out the bracketed terms $\{\dots\}$ to obtain

$$|SG_z^+\rangle = [\langle SG_x^+ | SG_z^+\rangle] |SG_x^+\rangle + [\langle SG_x^- | SG_z^+\rangle] |SG_x^-\rangle \quad (6.0.3)$$

where the quantities inside the square brackets are only (complex) numbers since these are “dot” products or projection between two (ket) vectors.

8. Now lets compare Eq. (6.0.3) with Eqs. (2.4.13) and (2.4.14) (page 23) which are reproduced below for your convenience:

$$|SG_x^+\rangle = \frac{1}{\sqrt{2}} [|SG_z^+\rangle + |SG_z^-\rangle] \quad (6.0.4)$$

and

$$|SG_x^-\rangle = \frac{1}{\sqrt{2}} [|SG_z^+\rangle - |SG_z^-\rangle] \quad (6.0.5)$$

We can add these two equations to obtain $|SG_z^+\rangle$ in terms of $|SG_x^+\rangle$ and $|SG_x^-\rangle$:

$$|SG_z^+\rangle = \frac{1}{\sqrt{2}} [|SG_x^+\rangle + |SG_x^-\rangle] \quad (6.0.6)$$

9. By comparison of Eq. (6.0.6) and Eq. (6.0.3) it follows that:

$$\langle SG_x^- | SG_z^+\rangle = \frac{1}{\sqrt{2}} \quad (6.0.7)$$

and

$$\langle SG_x^+ | SG_z^+\rangle = \frac{1}{\sqrt{2}} \quad (6.0.8)$$

Note that we could have obtained the above two equations directly by multiplying both sides of Eqs. (6.0.4) and (6.0.5) by $\langle SG_z^+ |$.

10. So what does all this mean? (Note: you already obtained the two equations above in a homework problem earlier and showed that the angle between these kets is $\pi/4$.)
11. Now go back and look at the results of your homework (item number 3 above) again. You got the probability of obtaining an $|SG_x^+\rangle$ or $|SG_x^-\rangle$ state as a result of the experiment are both equal to 1/2. (Dont take this statement for granted convince yourself by doing the experiment.)
12. Would it be correct to say that the probability of obtaining an $|SG_x^+\rangle$ state from a beam of $|SG_z^+\rangle$ waves is: $\{|\langle SG_x^+ | SG_z^+\rangle|\}^2$
13. Similarly the probability of obtaining an $|SG_x^-\rangle$ state from a beam of $|SG_z^+\rangle$ waves is: $\{|\langle SG_x^- | SG_z^+\rangle|\}^2$?

14. So the probability of obtaining these measurements is basically a dot product. The dot product can also be interpreted as a projection. Do you find this last statement to be meaningful?
15. Rationalize everything we are seeing here with respect to the plane polarized light analogy. Indeed the probability of obtaining an x' -polarized beam from an x -polarized beam is related to the “dot” product of x' onto x .
16. What is the problem? You have an SG_z^+ state coming in. Your measurement “changes” this state to either SG_x^+ or SG_x^- with equal probability. For example, using the x-magnetic field you can certainly not get SG_z^+ back or SG_y^+ for that matter. You can only see SG_x^+ or SG_x^- . This is what many people refer to as the “wavefunction collapses when you make a observation”. We have not used the terminology “wavefunction” before but this statement essentially means that the ket $|SG_z^+\rangle$ collapses to either $|SG_x^+\rangle$ or $|SG_x^-\rangle$ upon “measurement” using an x-magnetic field!!
17. These observations are profound and have long reaching implications in elucidating to us the strange dissimilarities between the quantum theory and the classical framework that we are so much used to. All this of course makes sense when you interpret states as vectors and measurements as “dot” products.
18. **Homework:** Now go ahead and do the experiment: Attach detectors to all outputs and

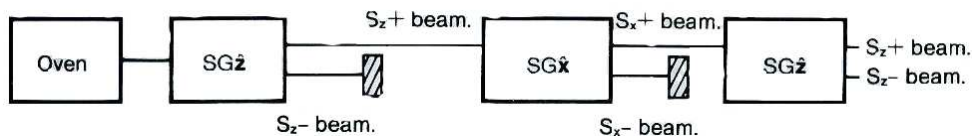


Figure 18:

explain the numbers you get using the “generalized-dot” products we have seen above.

19. Work out the above experiment for SG_y .
20. Replace SG_y with a magnetic field along an arbitrary direction (You can use the pull down menu to set angle θ and ϕ for the direction of an arbitrary magnetic field.) Plot the population on $|SG_z^+\rangle$ and $|SG_z^-\rangle$ as a function of θ and ϕ (Two three dimensional plots) and comment on what you see.