4.1 Spin matrices

1. Consider the *ket* vectors $|+\rangle$ and $|-\rangle$. Let these *ket* vectors represent the up-spin and down-spin states of an electron along the z-orientation. (*i.e.*, $|S_z^+\rangle$ and $|S_z^-\rangle$) A state with spin = +1/2 and is represented by the vector $|+\rangle$. What is meant by this statement is that $S_z|+\rangle = +\hbar(1/2)|+\rangle$. The state with spin = -1/2 is represented by the vector $|-\rangle$. Again, this statement implies $S_z|-\rangle = -\hbar(1/2)|-\rangle$) That is, these are *eigenstates* of the S_z operator. These two vectors form a 2-dimensional vector space that is *complete* and *orthonormal*. In matrix notation, these *ket* vectors may be written as

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.1.13}$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.14}$$

This is based on the isomorphism between $|+\rangle$ and x-polarized light and $|-\rangle$ and y-polarized light.

Since these two vectors form a 2-dimensional vector space that is *complete* and *orthonormal* the resolution of the identity in this space can be written as:

$$\{|+\rangle\langle+|\} + \{|-\rangle\langle-|\} = I \tag{4.1.15}$$

2. Using these *ket* vectors the S_z operator can be represented as follows:

$$\begin{aligned} S_z &\equiv S_z\left[\{|+\rangle\langle+|\} + \{|-\rangle\langle-|\}\right] \\ &= \left[\frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|\right] \\ &= \frac{\hbar}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right] \end{aligned} \tag{4.1.16}$$

(Note that the quantity in square brackets [...] on the left side in Eq. (4.1.16) is just the identity as per Eq. (4.1.15). Also note that we have used $S_z|+\rangle = +\hbar(1/2)|+\rangle$ and $S_z|-\rangle = -\hbar(1/2)|-\rangle$ to obtain Eq. (4.1.16). Obtain similar expressions for S_x and S_y.

3. S_z can then be written in matrix form using the *basis-ket* vectors $|+\rangle$ and $|-\rangle$ as:

$$\begin{aligned} S_z &\equiv \begin{pmatrix} \langle+|S_z|+\rangle & \langle+|S_z|-\rangle \\ \langle-|S_z|+\rangle & \langle-|S_z|-\rangle \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4.1.17}$$

Homework: Write down similar matrix forms for the expressions you obtained for S_x and S_y in the previous problems. These three matrices are called the Pauli-spin matrices.

4. **Homework: Using the three matrices you have for S_x, S_y, and S_z, confirm that these matrices do not commute.**

5. Pauli-spin matrices are 2×2 matrices. Which means they will act on 2×1 vectors. As noted earlier

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.1.18}$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.19}$$

And the Pauli-spin matrices can act on either these vectors or linear combinations of these vectors. Such vectors obtained from arbitrary linear combinations of $|+\rangle$ and $|-\rangle$ are called "spinors" (which comes from **spin**-vect**ors**. And in general the coefficients in front of each vector $|+\rangle$ and $|-\rangle$ can be complex.

