II Introductory Linear Algebra-I

1. The discussion will start from the very basics.

2. We will start with simple concepts from three-dimensional vector spaces.

3. In 3 D a vector has three components:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{II.0.1}$$

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the x, y and z directions respectively. $\hat{i}$, $\hat{j}$ and $\hat{k}$ may also be called basis vectors or just bases and this is a terminology that we will use often. Vectors can be represented in the following form:

$$\vec{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{II.0.2}$$

4. The vector is fully described by: (i) its magnitude, and (ii) its direction. Compare this with a scalar number that only has a magnitude.

5. The magnitude of the vector $\vec{r}$ can be calculated by performing the following mathematical operation:

$$|\vec{r}| \equiv \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{|x|^2 + |y|^2 + |z|^2} \tag{II.0.3}$$

where we have introduced the definition of the "dot" product of two vectors:

$$\begin{aligned} \vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ \vec{b} &= b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \\ \vec{a} \cdot \vec{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned} \tag{II.0.4}$$

6. The "dot" product is a very fundamental concept, and very useful for our further development hence let us look at it a little closely.

7. Note that by the definition of matrix multiplication, I can write the dot product of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} = (a_1 \ a_2 \ a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{II.0.5}$$

Can you confirm this using the laws of matrix multiplication? Talk to me if you have a problem here.

8. **Homework: Multiply the two matrices:**

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} and B = \begin{pmatrix} 1 & 4 & 2 & 5 \\ 3 & 6 & 9 & 10 \end{pmatrix} \tag{II.0.6}$$

Pay careful attention to the order in which these matrices are multiplied. What is A multiplied by B? What is B multiplied by A? Based on this exercise what can you say about the order in which matrices are multiplied?

II.1 The Dual Space

9. The object $(a_1\ a_2\ a_3)$ in Eq. (II.0.5) can be referred to as the *dual-space analogue* of the object:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{II.1.7}$$

(Thing "kets" and "bras".)

10. Note that every vector has its *dual-space analogue*, and for real numbers, the dual space analogue is just the transpose of the vector.

11. So we may write:

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} \tag{II.1.8}$$

12. However, in general the components of the vector $\vec{a}$ need not be real (they could be complex!!) and then the dual space analogue is not simply the transpose. In the complex case the dual space analogue of the vector $\vec{a}$ is defined as:

$$\vec{a}^\dagger = (a_1^*\ a_2^*\ a_3^*) \tag{II.1.9}$$

where the "superscripted *" imply *complex conjugation*. (Do you know what that means? If not ask the instructor and he will explain it to you.)

13. The dual space analogue is a very important concept in quantum theory, as we will see later. For now we understand the dual space analogue is the complex conjugated transpose of a vector.

14. The basis vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$ that we introduced earlier, have a very important property and that is the "dot" product of each of these with itself is equal to 1, but the the dot product of one with any of the others is always 0 (zero). This means:

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 \tag{II.1.10}$$

but

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \tag{II.1.11}$$

These relations make the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ an *ortho-normal* basis set. The term *ortho-normal* has two parts: the first part *ortho* is part of the word *orthogonal*. Two vectors are said to be *orthogonal* when they are at 90 degrees. (Note that the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are at 90 degrees to each other, as is required from the fact that these are unit vectors along the x, y and z directions.) The second part of *ortho-normal* is the word *normal*: a vector is normalized when its magnitude is 1.

Do you see connections to the PIB states and see why vectors might be useful in understanding quantum theory?

III Introductory Linear Algebra-II: *Dyads*

1. We shall now introduce a very important mathematical quantity called a *dyad*. This is fundamental in the theory of linear vector spaces, and turns out to be of great importance in quantum mechanics. (Make sure you got the exercise on matrix multplication in the previous section before you proceed into this one.)

    - Before we introduce the dyad, lets consider the dot product in Eq. (II.0.5). The product of a (1×3) matrix with a (3×1) matrix leads to a (1×1) (*i.e.* a scalar number) matrix.
    - What would happen if we had things backwards?

    $$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} (a_1 \; a_2 \; a_3) \tag{III.0.1}$$

    This represents a (3×1) matrix multiplied with a (1×3) matrix and hence the result should be a (3×3) matrix !!
    - The quantity in Eq. (III.0.1) is called the *dyadic product* or *outer product* of the two vectors. By contrast the dot product in Eq. (II.0.5) is also known as the *inner product* of two vectors.

2. To understand some of the implications of the dyadic product (there will be other implications that we will realize when we introduce the Dirac notation) let us consider the basis vectors in 3-dimensional space:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{III.0.2}$$

3. The dyadic product:

$$\vec{i}\vec{i}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{III.0.3}$$

that is a matrix. Similarly the dyadic products $\vec{i}\vec{i}^{\dagger}$, $\vec{i}\vec{j}^{\dagger}$, $\vec{i}\vec{k}^{\dagger}$, $\vec{j}\vec{i}^{\dagger}$, $\vec{j}\vec{j}^{\dagger}$, $\vec{j}\vec{k}^{\dagger}$, $\vec{k}\vec{i}^{\dagger}$, $\vec{k}\vec{j}^{\dagger}$, $\vec{k}\vec{k}^{\dagger}$ are all matrices with 1 at one of the positions in the matrix and zeroes everywhere else.

4. **Homework, not to turn in but do talk to us if you have trouble:**

   (a) **Write down all the dyadic products $\vec{i}\vec{i}^{\dagger}$, $\vec{i}\vec{j}^{\dagger}$, $\vec{i}\vec{k}^{\dagger}$, $\vec{j}\vec{i}^{\dagger}$, $\vec{j}\vec{j}^{\dagger}$, $\vec{j}\vec{k}^{\dagger}$, $\vec{k}\vec{i}^{\dagger}$ $\vec{k}\vec{j}^{\dagger}$, $\vec{k}\vec{k}^{\dagger}$**

   (b) **Homework: Show that (to be turned in):**

$$\vec{i}\vec{i}^{\dagger} + \vec{j}\vec{j}^{\dagger} + \vec{k}\vec{k}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{III.0.4}$$

   The matrix on the right hand side in this equation is called the identity or unit matrix. This relation is called the *resolution of the identity* and we will have a great deal of need for this expression.

   (c) Using the results obtained above, show that the matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \tag{III.0.5}$$

   can we written as a linear combination $\vec{i}\vec{i}^{\dagger}$, $\vec{i}\vec{j}^{\dagger}$, $\vec{i}\vec{k}^{\dagger}$, $\vec{j}\vec{i}^{\dagger}$, $\vec{j}\vec{j}^{\dagger}$, $\vec{j}\vec{k}^{\dagger}$, $\vec{k}\vec{i}^{\dagger}$ $\vec{k}\vec{j}^{\dagger}$, $\vec{k}\vec{k}^{\dagger}$.

   What is the linear combination?

5. Note: You have just performed a very important exercise. An important essence of quantum mechanics is in this exercise.

6. Firstly, the resolution of the identity is also known as the completeness relation. Any vector can be written as a linear combination of an *ortho-normal* and *complete* set of vectors. As the set $\hat{i}$, $\hat{j}$ and $\hat{k}$ are *ortho-normal* (as seen in point 14 of Section II) and *complete* (as seen in point 4b) any vector in three-dimension can be written as a linear combination of these vectors.

7. Secondly, we also note from point 4 that any matrix is writable as a linear combination of the *dyadic*-basis: $\vec{i}\vec{i}^{\dagger}$, $\vec{i}\vec{j}^{\dagger}$, $\vec{i}\vec{k}^{\dagger}$, $\vec{j}\vec{i}^{\dagger}$, $\vec{j}\vec{j}^{\dagger}$, $\vec{j}\vec{k}^{\dagger}$, $\vec{k}\vec{i}^{\dagger}$ $\vec{k}\vec{j}^{\dagger}$, $\vec{k}\vec{k}^{\dagger}$.

