5 A brief note on change of basis

1. We spoke about the definition of a wavefunction. It is the projection of the abstract "ket" $|\psi\rangle$ onto the coordinate representation "ket", *i.e.*, $\langle x|\psi\rangle \equiv \psi(x)$, the wavefunction.

2. But then the abstract "ket" $|\psi\rangle$ can be represented using any *complete set of kets*, and we have chosen above to use the coordinate representation, which we know to be complete from the identity:
$$\int dx\, |x\rangle \langle x| = 1 \tag{5.0.1}$$
But we could have chosen to use some other set, for example the momentum representation, since:
$$\begin{aligned} |\psi\rangle &\equiv \int dx\, |x\rangle \langle x|\psi\rangle = \int dx\, |x\rangle\, \psi(x) \\ &\equiv \int dk\, |k\rangle \langle k|\psi\rangle = \int dk\, |k\rangle\, \tilde{\psi}(k) \\ &\equiv \sum_i |i\rangle \langle i|\psi\rangle = \sum_i |i\rangle\, c_i. \end{aligned} \tag{5.0.2}$$
where $\psi(x)$, $\tilde{\psi}(k)$ and c_i are just projects of the abstract "ket" $|\psi\rangle$ onto the "axes" of the chosen representation.

3. The statement along with our discussion on measurement as a "dot" product may be viewed to imply that, in fact, it is the projection of $|\psi\rangle$ in some basis that is measured (or calculated) and from such a projection we (sometimes) struggle to recreate the entire "ket" $|\psi\rangle$.

4. That begs the question, how do these projections relate to each other. That is how is $\psi(x)$ related to $\tilde{\psi}(k)$ and c_i and vice versa.

5. This is the question we address here as part of discussion on "change of basis".

6. Consider the following sequence of arguments:
$$\begin{aligned} \psi(x) &= \langle x|\psi\rangle \\ &= \langle x\, |I|\, \psi\rangle \\ &= \left\langle x \left| \int dk\, |k\rangle \langle k| \right| \psi \right\rangle \end{aligned} \tag{5.0.3}$$
where we have inserted the identity in terms of the momentum representation. Now, since $\langle x|$ is a vector and $\int dk$ is essentially a continuous summation, we can take the vector inside the summation (to be discussed in class), leading to:
$$\psi(x) = \int dk\, \langle x|k\rangle \langle k\, ||\, \psi\rangle \tag{5.0.4}$$
Now, $\langle x|k\rangle$ is the projection (or representation) of the momentum eigenstate in the coordinate representation. That is, $\langle x|k\rangle$ is a function of x, that is also an eigenstate of the momentum operator.

7. So, what is $\langle x|k\rangle$? Well, $\langle x|k\rangle = \exp(\imath kx)$, because $\exp(\imath kx)$ is an eigenstate of the momentum operator and is a function of x. Read the previous point again to make sure you understand this.

8. That leads to:

$$\begin{aligned} \psi(x) &= \int dk \exp(\imath kx) \langle k || \psi\rangle \\ &= \int dk \exp(\imath kx) \tilde{\psi}(k) \end{aligned} \tag{5.0.5}$$

9. Some of you might recognize that the above equation is a Fourier transform. Indeed, one needs to construct a Fourier transform to convert a wavefunction into its momentum space analogue and vice versa. But this is now an example of a change of basis. That is if we have $\tilde{\psi}(k)$, we can always use the above equation to get $\psi(x)$, that is the projection of $|\psi\rangle$ on to a different basis.

10. **Homework: Work out the transformation of $\psi(x)$ onto some discrete basis $\{|i\rangle\}$.**

