5. **Simultaneous eigenstates**: Consider two operators that commute:

$$\left[\hat{A}, \hat{B}\right] = 0 \tag{J.28}$$

Let $\hat{A}$ satisfy the following eigenvalue equation:

$$\hat{A}\left|\eta\right\rangle = a\left|\eta\right\rangle \tag{J.29}$$

Multiplying both sides by $\hat{B}$

$$\hat{B}\left[\hat{A}\left|\eta\right\rangle\right] = \hat{B}\left[a\left|\eta\right\rangle\right] = a\hat{B}\left|\eta\right\rangle = a\left|\chi\right\rangle \tag{J.30}$$

where we have assumed $\hat{B}\left|\eta\right\rangle \equiv \left|\chi\right\rangle$.

Now lets try and use the commutation relation:

$$\hat{B}\left[\hat{A}\left|\eta\right\rangle\right] = \hat{A}\left[\hat{B}\left|\eta\right\rangle\right] = \hat{A}\left|\chi\right\rangle \tag{J.31}$$

From the right hand sides of the last two equations it follows that $\left|\chi\right\rangle$ is also an eigenvector of $\hat{A}$ with eigenvalue a. This could only happen if:

(a) $\left|\chi\right\rangle = b\left|\eta\right\rangle$ since $\left|\eta\right\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue a. Hence commuting operators have *simultaneous* eigenstates. That is these can be exactly measured simultaneously. (By extension two operators that do not commute cannot be measured simultaneously as we will see in the next section.) This is a very important distinguishing factor with respect to classical mechanics. In classical mechanics you can measure any two observables simultaneously. In quantum mechanics, only variables whose (Hermitian) operators commute can be observed simultaneously. Now go back and read the Stern-Gerlach experiment. It must mean that since S_z, S_x and S_y cannot be observed simultaneously, *the operators corresponding to these observables, the Pauli spin matrices, do not commute*.

(b) There is a second possibility and that is $\left|\chi\right\rangle$ is not equal to a constant times $\left|\eta\right\rangle$. In this case the operator $\hat{A}$ must have degenerate states ($\left|\chi\right\rangle$ and $\left|\eta\right\rangle$) and the operator $\hat{B}$ can be used to obtain all the degenerate states of $\hat{A}$. But even in this case, the non-degenerate eigenvectors of $\hat{A}$ are *simultaneously* eigenstates of $\hat{B}$ as is clear from the previous case.

6. Actually case (b) is also deeply connected with Stern-Gerlach as we will see later.

7. **Expectation or Average value of an operator**

If the $\left|\psi\right\rangle$ represents the state of a system, the *expectation value* of an operator $\hat{A}$ with respect to that state is given by the quantity

$$\left\langle\hat{A}\right\rangle = \left\langle\psi\right|\hat{A}\left|\psi\right\rangle \tag{J.32}$$

What does this mean? If there is an observable associated with the operator $\hat{A}$, then $\left\langle\hat{A}\right\rangle$ is **the average measured value of the observable.**

(a) Case 1: The state vector ψ is an eigenstate of the operator $\hat{A}$. In that case, if a is the associated eigenvalue then $\left\langle \hat{A} \right\rangle = \langle\psi|\, \hat{A}\, |\psi\rangle = a$. (Note: $\langle\psi|\ \psi\rangle = 1$.) That is the observable has a measured value equal to the eigenvalue of the operator $\hat{A}$. Consider Figure 7 (reproduced below) as a way to understand this concept. Here, an S_z^+ state is sent into an SGz measuring device and the resultant measurement is the eigenvalue of the S_z^+ state with respect to the $\hat{S}_z$ operator. Hence if ψ is an eigenstate of the operator, the corresponding measured value, or expectation value is a,

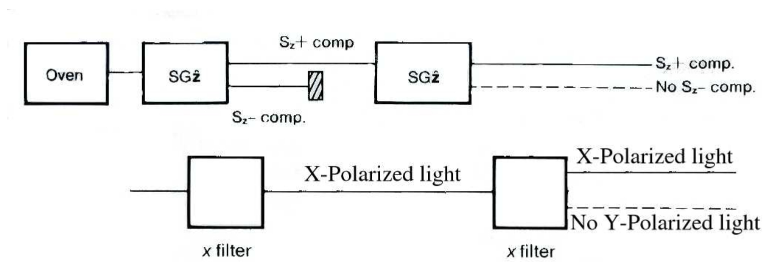


Figure 19:

(b) Case 2: The state vector ψ is not an eigenstate of the operator $\hat{A}$. In this case, if $\hat{A}$ is a Hermitian operator then the *eigenstates of a Hermitian operator form a complete ortho-normal set*. **Hence any vector like ψ can be expanded as a linear combination of such a complete set of vectors.** Therefore, if $\{\chi_i\}$ are the family of eigenvectors of $\hat{A}$, with eigenvalues $\{a_i\}$, that is

$$\hat{A}\chi_i = a_i\chi_i \qquad i = 1, 2, 3, \cdots \tag{J.33}$$

then we can expand the vector ψ as a linear combination of the vectors $\{\chi_i\}$, as

$$|\psi\rangle = \left[\sum_i |\chi_i\rangle \langle\chi_i|\right] |\psi\rangle = \sum_i |\chi_i\rangle \langle\chi_i|\, |\psi\rangle \tag{J.34}$$

Note we have used the "resolution of identity" here. If we denote the number $\langle\chi_i|\,|\psi\rangle = c_i$ (note that this is a "dot" product and hence is a number), then

$$|\psi\rangle = \sum_i c_i\, |\chi_i\rangle \tag{J.35}$$

The expectation value of the operator $\hat{A}$ can then be written as

$$\begin{aligned} \left\langle \hat{A} \right\rangle = \langle\psi|\, \hat{A}\, |\psi\rangle &= \left\{\sum_j c_j{}^* \langle\chi_j|\right\} \hat{A} \left\{\sum_i c_i\, |\chi_i\rangle\right\} \\ &= \sum_j \sum_i \left[c_j{}^* \langle\chi_j|\right] \hat{A} \left[c_i\, |\chi_i\rangle\right] \\ &= \sum_j \sum_i c_j{}^* c_i \langle\chi_j|\, \hat{A}\, |\chi_i\rangle \end{aligned} \tag{J.36}$$

Note that $\hat{A}\,|\chi_i\rangle = a_i\,|\chi_i\rangle$ and since $\langle\chi_j|\,|\chi_i\rangle = \delta_{i,j}$ (since the eigenfunctions of $\hat{A}$ are orthonormal!!) we can simplify the above equation to

$$\begin{aligned} \left\langle \hat{A} \right\rangle &= \sum_j \sum_i c_j{}^* c_i a_j \langle\chi_j|\,|\chi_i\rangle \\ &= \sum_j \sum_i c_j{}^* c_i a_i \delta_{i,j} \end{aligned} \tag{J.37}$$

The last term has two summations. The summation over j yields a non-zero value onlu when $j = i$ on account of the Kronecker delta: $\delta_{i,j}$. Hence we have

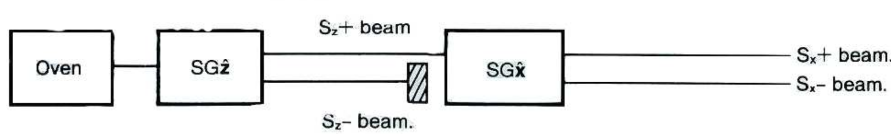
$$\left\langle \hat{A} \right\rangle = \sum_i c_i{}^* c_i a_i = \sum_i N_i a_i \tag{J.38}$$

where we have just substituted $N_i = c_i{}^* c_i$.

Now the last equation does look like an average of all the a_i values. (Why? Note that $c_i{}^* c_i \leq 1$ because ψ should be normalized and that implies $\sum_i c_i{}^* c_i = 1$ from Eq. (J.35).)

When you perform a large number of measurements, the average of all those measurements will be $\left\langle \hat{A} \right\rangle$. That is why it is called the "expectation value". But a single measurement will yield a value corresponding to one of the eigenstates of $\hat{A}$!!! The average of all these measurements, just like in Eq. (J.38), will give $\left\langle \hat{A} \right\rangle$.

Consider the SG experiments and see if you can rationalize this fact. Especially consider the experiment



You performed this experiment using the java script. Lets now look at the average value of S_x after the second measurement. When you conduct a large number of experiments (say 10,000 runs), you get an almost equal amount of S_x^+ and S_x^- and the average or expectation value turns out to be zero. (We rationalized based on the projection of the S_z^+ state on S_x^+ and S_x^- that indeed we should get equal amounts of both.)

However, when you perform just one experiment you get either S_x^+ or S_x^-. Hence, the statement above, a single measurement will yield a value corresponding to one of the eigenstates of $\hat{A}$!!! The average of all these measurements, just like in Eq. (J.38), will give $\left\langle \hat{A} \right\rangle$.

