D Introduction to Representation Theory

D.1 Introductory Linear Algebra

1. Revise earlier linear algebra sections and make sure you did the following homework problems

   (a) **Write down all the dyadic products** $\vec{i}\vec{i}^\dagger$**,** $\vec{i}\vec{j}^\dagger$**,** $\vec{i}\vec{k}^\dagger$**,** $\vec{j}\vec{i}^\dagger$**,** $\vec{j}\vec{j}^\dagger$**,** $\vec{j}\vec{k}^\dagger$**,** $\vec{k}\vec{i}^\dagger$ $\vec{k}\vec{j}^\dagger$**,** $\vec{k}\vec{k}^\dagger$

   (b) **Homework: Show that:**

$$\vec{i}\vec{i}^\dagger + \vec{j}\vec{j}^\dagger + \vec{k}\vec{k}^\dagger = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \tag{D.1}$$

   The matrix on the right hand side in this equation is called the identity or unit matrix. This relation is called the *resolution of the identity* and we will have a great deal of need for this expression.

2. The resolution of the identity is also known as the completeness relation.

3. Any vector can be written as a linear combination of an *ortho-normal* and *complete* set of vectors. As the set $\hat{i}$, $\hat{j}$ and $\hat{k}$ are *ortho-normal* and *complete* (as seen in point 1b) it comes as no surprise to us that any vector in three-dimension can be written as a linear combination of these vectors.

4. **Homework:** Show that the spin states SG_z^+ and SG_z^- (we learn a better way to represent these states soon) form a complete orthonormal set. (Use the analogy to the x- and y-filters.) Hint: This question is simpler than you might think!!

D.2 Dirac Notation

1. The discussion in the linear algebra handouts is heavily couched in the 3-dimensional space. However, you will note that there is no restriction as such, on the algebra. All of what we had learned (dot products, dyads, resolution of identity, etc.) could very well be true for vectors in say 4-dimensions (or (4×1) matrices and their (1×4) dual counterparts) and matrices in 4-dimensions (*i.e.* (4×4) matrices). It would be very difficult (!!!!) for me to draw something in 4-dimensions, but in this lecture we will see how we can think in 4 or more dimensions.

2. While constructing the *analogy of the Stern-Gerlach spin states to polarized light* we learned that the polarized light can be treated using vectors and the Stern-Gerlach states are analogous to these. In a quantum mechanical system there will in general be a large number of such states. There should be a way for us to generalize the theory of vectors to arbitrary dimensions. This is what we will look into next.

3. Why arbitrary dimensions? We already saw that *four real dimensions* or two complex dimension were required to represent the spin states. See page 27 (the Stern Gerlach handout).

4. To make it to n-dimensions, the first thing we will do is introduce a new set of notations. The notation in the previous subsection is very nice for 3-dimensional space but turns out to be highly cumbersome in n-dimensions. The notation we are about to introduce is that due to Dirac, and hence is called the Dirac notation.

   - A vector in n-dimensions will be represented by the object: $|i\rangle$, and we will call this a *ket*, or a *ket*-vector.
   - The dual space analog of the *ket*, $|i\rangle$, is called the *bra* and is represented by $\langle i|$. (**Note: The *ket*, $|i\rangle$, is a vector in n-dimensions. A corresponding vector in 3-dimensions we represented as $\vec{i}$, earlier. The *bra*, $\langle i|$, is the dual-space analogue of the *ket*. In 3-dimensions we represented the dual space analogue of $\vec{i}$ as $\vec{i}^\dagger$. It is useful to keep these straight in our mind, so we do not get lost while we walk up to n-dimensions.**)
   - The reason for these peculiar names (*bra* and *ket*) is, the "dot" product is now a product of the *bra* and the *ket* and hence is the *bra-ket* :-) $\langle i| \, |i\rangle$, analogous to $\vec{i}^\dagger\vec{i}$ in 3-dimensions, which is a number like $\langle i| \, |i\rangle$. (In practice we choose to be lazy and drop one of the vertical bars in the middle of the *bra-ket* and represent it as $\langle i| \, i\rangle$.)
   - The outer product or the dyadic product that we made so much fuss about in the previous subsection if a *ket* $\times$ a *bra*. **Why?** Because the dyadic product in 3-dimensions (see Eq. (B.1)) is a vector times its dual space analogue, *for example* $\vec{i}\vec{k}^\dagger$.
   - **And the vector in n-dimensions is a *ket*, and its dual space analogue is the *bra*.** So the dyadic product is $|i\rangle \langle i|$.

5. This notation is due to P. A. M. Dirac in the late 1920s and hence is called the Dirac notation.

6. The resolution of identity in Dirac's notation.

   - Do we recall what a complete set of vectors in 3-dimensions is? A set of vectors that are *ortho-normal* and a set of vectors that obey the resolution of identity, Eq. (D.1). So, as we saw in the previous section, $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a complete set.
   - In more simple terms, what we mean by $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a complete set in 3-dimensions is that any vector in three-dimensions can be written as a linear combination (as in Eq. (A.1)) of $\hat{i}$, $\hat{j}$ and $\hat{k}$. (**Homework: By extension, could you comment on why Eqs. 2.9, 2.10 2.13, and (2.14), might actually make sense based on what we have noted here? Utilize your result from the homework problem on page 32.**)
   - Here we want to see what the corresponding set of rules would be for n-dimensions. And as we will see below these set of rules are very similar to what we have in 3-dimensions.
   - But, first, what does the number n in $\{|n\rangle\}$ mean? It could, for example, be a numbering scheme:

$$|m\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{D.2}$$

     where the 1 is on the m-th position. n columns in all. So $|m\rangle$ in the equation above is a ($n\times$ 1) matrix that has all zeroes except at the m-th position.
   - **Note: Eq. (D.2) is just an example of what $|m\rangle$ can be. More generally, it is just some vector in n-dimensions.**
   - Now we are in a position to generalize the resolution of identity or completeness relation that we derived earlier for the 3-dimensional case. The corresponding relation for n-dimensions is:

$$\sum_{i=1}^{n} |i\rangle \langle i| = I \tag{D.3}$$

     where the right hand side is the identity matrix. You can see that for n=3, the left hand side in Eq. (D.3) has three terms just like in Eq. (D.1). *In fact Eq. (D.3) is identical to Eq. (D.1) in three-dimensions* !!! In higher dimensions Eq. (D.3) provides us an additional tool.

- **Homework:**
  (a) Using the definition for the ket in Eq. (D.2) show that Eq. (D.3) is valid of n=4. (Hint: You have already shown this to work for n=3 in the previous homework. Do the same thing you did in the previous homework, but now for 4-dimensions. That is the vectors would be (4 × 1) and the matrices would be (4 × 4)).
  (b) Can you see on the basis of this homework that Eq. (D.2) would be valid for any n? If not you are welcome to come see me.
  (c) In an earlier homework problem, you showed how $|SG_z^+\rangle$ and $|SG_z^-\rangle$ (notice the new *ket* notation being used here to describe these states) form a complete set. Can you show that $|SG_x^+\rangle$ and $|SG_x^-\rangle$ form a complete set as well? How about $\left|SG_y^+\right\rangle$ and $\left|SG_y^-\right\rangle$? What does all this tell you?

7. The n-dimensional space that we just spoke about, is also called the **Hilbert space** in quantum mechanics. (After D. Hilbert a famous mathematician in the 20 century.)

8. Why do we care about all this?

9. These are the necessary mathematical tools we need to develop quantum mechanics.

10. Consider the Stern-Gerlach experiments we studied earlier. We may now choose to the denote the SG_z^+ state by the *ket* $|SG_z^+\rangle$. Why is this interesting? Because now we see that a state in quantum mechanics such as $|SG_z^+\rangle$ is essentially a vector in some n-dimensional space (the Hilbert space).

11. And since $|SG_z^+\rangle$ is a *ket*, it can be written as a linear combination of some complete set of *kets* like in Eq. (D.2). That is,

$$\left|SG_z^+\right\rangle \equiv \left\{\sum_{i=1}^{n} |i\rangle \langle i|\right\} \left|SG_z^+\right\rangle = \sum_{i=1}^{n} |i\rangle \langle i| \left. SG_z^+\right\rangle = \sum_{i=1}^{n} c_i |i\rangle \tag{D.4}$$

where $c_i = \langle i| SG_z^+\rangle$ are just numbers (*but they could be complex* !!). Note, we have introduced "1", *i.e.* the resolution of the identity of Eq. (D.3) in the second part of Eq. (D.4). Physically c_i are the "dot" product or inner product or *bra-ket* of $\langle i|$ with $|SG_z^+\rangle$. The c_i are just numbers and note the similarity between the above equation and the first equation in the linear algebra handout Eq. (A.1). x, y and z are also just numbers in Eq. (A.1).)

12. In addition, Eq. (D.4) tells us how to "change basis" from $|SG_z^+\rangle$ to $\{|i\rangle\}$. Can we visualize change of basis in three-dimensions?

