E The Position and the momentum representation and the Wavefunction

1. We shall also note here that the set $\{|n\rangle\}$ represented in Eq. (D.3) is a discrete set. How do we know this is discrete, the summation in Eq. (D.3) has a countable number of terms. In three-dimensional the summation in Eq. (D.3) has three terms; in four-dimensions it has four terms and in n-dimensions the summation in Eq. (D.3) has n terms. In the next section we will discuss a continuous representation which is basically obtained by converting the summation in Eq. (D.3) into an integral:

$$\sum \rightarrow \int \tag{E.5}$$

At this point it will be useful to review some of your calculus. In particular we would like to remember that the integration is "the limit of a sum". Hence the integration is very similar to a sum, but only has infinitely many terms in it. Hence the correspondence in Eq. (E.5) makes sense.

2. The eigenstates of momentum for a *continuous* representation which we discussed earlier (Eq. (E.5).

$$\int dk\, |k\rangle \langle k| = 1 \tag{E.6}$$

Why continuous? The k in Eq. (H.9) can take on any real value and $\exp\{\imath kx\}$ would still remain an eigenstate of the momentum operator.

3. Eigenstates of many different kinds of "special" operators in quantum mechanics always form a complete set. We will prove this general statement in detail later in this class.

4. Like the momentum operator, there is another kind of operator in quantum mechanics called the position operator.

$$\hat{x}\, |x\rangle = x\, |x\rangle \tag{E.7}$$

The eigenstates of the position operator form another important complete set of *ket* vectors that form a continuous representation.

$$\int dx\, |x\rangle \langle x| = 1 \tag{E.8}$$

5. As the name suggests, the variable "x" above is the position (in 3-dimensions or in n-dimensions, but it is easier to picture this in 3D). What this means all point in a 3-dimensional space (for example) form a complete set of *ket* vectors. (This point is *extremely* subtle.)

6. **The Wavefunction:** In the Stern-Gerlach experiments we represented the states using the *ket* $|SG_x\rangle$. More generally, the state of any system can be represented by a *ket*, say $|\psi\rangle$. Consider the inner product of the *bra* state $\langle x|$ with a *ket* vector $|\psi\rangle$, *i.e.* $\langle x\, |\psi\rangle \equiv \psi(x)$. This quantity is called the wavefunction. Hence the wavefunction is the inner product of

the abstract *ket* vector that represents the state of the system (for example the state of the Stern-Gerlach experiment) with the position representation. We will discuss a lot more in the next few lectures regarding this "wavefunction".

7. In fact the story of quantum mechanics, as we are going to learn it, is the story of how to find the wavefunction of the system. Why is this important?

   (a) We noted that the wavefunction is obtained by the inner product of the abstract *ket* vector that represents the state of the system with the position representation. (This process of performing this inner product is also called a *projection*. Hence, the wavefunction is the projection of the abstract *ket* vector $|\psi\rangle$ on to the position representation.)

   (b) Since $|\psi\rangle$ represents the state of the system, (as the states in the Stern-Gerlach experiment fully represent the state of the system, in a similar fashion $|\psi\rangle$ contains all information about the system). we would like to know everything there is to know about $|\psi\rangle$.

   (c) What is the equation that gives us $|\psi\rangle$? It is called the Schrödinger Equation, which we will see soon.

   (d) **Properties of the Wavefunction:** We will simply state the required properties here. Later when we solve our first quantum mechanical problem (the particle in a box) we will see how these properties become necessary.

   - The Wavefunction must be continuous.
   - The wavefunction must have finite values in all space.
   - The wavefunction must be normalized. That is the integral of the square of the wavefunction over all space must be 1:

   $$\langle \psi \mid \psi \rangle = \langle \psi | \left\{ \int dx \, | x \rangle \langle x | \right\} | \psi \rangle = \int dx \psi^*(x) \psi(x) = 1 \tag{E.9}$$

     This condition is extremely important, mathematically. It allows only a certain kind of function to be a wavefunction: ones that are *square integrable*. It also implies that the length of the ket $|\psi\rangle$ is always 1.
   - And finally the quantity $dx\psi^*(x)\psi(x) \equiv dx|\psi(x)|^2$ is interpreted as the probability density of the system. That is the probability of finding the system in a infinitesimal area of size dx around the point x.