

## 4 A Measurement is a Projection or a “dot” product (or inner product)!!

- Let us go back and reconsider the Stern Gerlach experiment that we studied earlier:

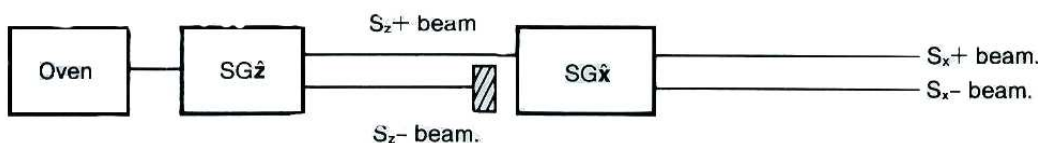


Figure 12:

- After the second magnetic field, the molecules can only be seen to have an  $S_x^+$  and  $S_x^-$  component.
- As a homework convince yourself by running the java script on the course website that the probability of obtaining an  $|SG_x^+\rangle$  or  $|SG_x^-\rangle$  state as a result of the above experiment are both equal to 1/2.
- Now to understand this mathematically consider the following:

$$|SG_z^+\rangle \equiv \left\{ \sum_{i=1}^n |i\rangle \langle i| \right\} |SG_z^+\rangle = \sum_{i=1}^n c_i |i\rangle \quad (4.1)$$

We have of course seen this equation before. All we have done is “resolve the identity” in terms of the kets  $\{|i\rangle\}$ . But we could have equally well done the following:

$$|SG_z^+\rangle \equiv \{ |SG_x^+\rangle \langle SG_x^+| + |SG_x^-\rangle \langle SG_x^-| \} |SG_z^+\rangle \quad (4.2)$$

Notice that the quantity inside  $\{\dots\}$  is just equal to 1. (The resolution of the identity!!!)

- How do we rationalize the above statement? We have noticed that the  $\hat{x}'$  and  $\hat{y}'$  vectors are “isomorphic” to the kets  $|SG_x^+\rangle$  and  $|SG_x^-\rangle$ . And any vector in the x-y plane can be written as a linear combination of the  $\hat{x}'$  and  $\hat{y}'$  vectors. Hence these vectors form a *complete set* in “2-dimensions”. The resolution of the identity is simply a mathematical language that conveys the same meaning as the above set of words.
- As a result of the above it must follow that  $|SG_x^+\rangle$  and  $|SG_x^-\rangle$  also form a complete set and hence the resolution of the identity in terms of these is valid.
- We can simplify Eq. (4.2) by multiplying out the bracketted terms  $\{\dots\}$  to obtain

$$|SG_z^+\rangle = [ \langle SG_x^+ | SG_z^+ \rangle ] |SG_x^+\rangle + [ \langle SG_x^- | SG_z^+ \rangle ] |SG_x^-\rangle \quad (4.3)$$

where the quantities inside the square brackets are only (complex) numbers since these are “dot” products or projection between two (ket) vectors.

8. Now lets compare Eq. (4.3) with Eqs. (2.9) and (2.10) which are reproduced below for your convenience:

$$|SG_x^+\rangle = \frac{1}{\sqrt{2}} [|SG_z^+\rangle + |SG_z^-\rangle] \quad (4.4)$$

and

$$|SG_x^-\rangle = \frac{1}{\sqrt{2}} [|SG_z^+\rangle - |SG_z^-\rangle] \quad (4.5)$$

We can add these two equations to obtain  $|SG_z^+\rangle$  in terms of  $|SG_x^+\rangle$  and  $|SG_x^-\rangle$ :

$$|SG_z^+\rangle = \frac{1}{\sqrt{2}} [|SG_x^+\rangle + |SG_x^-\rangle] \quad (4.6)$$

9. By comparison of Eq. (4.6) and Eq. (4.3) it follows that:

$$\langle SG_x^- | SG_z^+ \rangle = \frac{1}{\sqrt{2}} \quad (4.7)$$

and

$$\langle SG_x^+ | SG_z^+ \rangle = \frac{1}{\sqrt{2}} \quad (4.8)$$

Note that we could have obtained the above two equations directly by multiplying both sides of Eqs. (4.4) and (4.5) by  $\langle SG_z^+ |$ .

10. So what does all this mean? (Note: you already obtained the two equations above in a homework problem earlier and showed that the angle between these kets is  $\pi/4$ .)
11. Now go back and look at the results of your homework (item number 3 above) again. You got the probability of obtaining an  $|SG_x^+\rangle$  or  $|SG_x^-\rangle$  state as a result of the experiment are both equal to 1/2. (Dont take this statement for granted convince yourself by doing the experiment.)
12. Would it be correct to say that the probability of obtaining an  $|SG_x^+\rangle$  state from a beam of  $|SG_z^+\rangle$  waves is:  $\{|\langle SG_x^+ | SG_z^+ \rangle|\}^2$
13. Similarly the probability of obtaining an  $|SG_x^-\rangle$  state from a beam of  $|SG_z^+\rangle$  waves is:  $\{|\langle SG_x^- | SG_z^+ \rangle|\}^2$  ?
14. So the probability of obtaining these measurements is basically a dot product. The dot product can also be interpreted as a projection. Do you find this last statement to be meaningful?
15. Rationalize everything we are seeing here with respect to the plane polarized light analogy. Indeed the probability of obtaining an  $x'$ -polarized beam from an  $x$ -polarized beam is related to the “dot” product of  $x'$  onto  $x$ .

16. What is the problem? You have an  $SG_z^+$  state coming in. Your measurement “changes” this state to either  $SG_x^+$  or  $SG_x^-$  with equal probability. For example, using the x-magnetic field you can certainly not get  $SG_z^+$  back or  $SG_y^+$  for that matter. You can only see  $SG_x^+$  or  $SG_x^-$ . This is what many people refer to as the “wavefunction collapses when you make a observation”. We have not used the terminology “wavefunction” before but this statement essentially means that the ket  $|SG_z^+\rangle$  collapses to either  $|SG_x^+\rangle$  or  $|SG_x^-\rangle$  upon “measurement” using an x-magnetic field!!
17. These observations are profound and have long reaching implications in elucidating to us the strange dissimilarities between the quantum theory and the classical framework that we are so much used to. All this of course makes sense when you interpret states as vectors and measurements as “dot” products.
18. **Homework:** Now go ahead and do the experiment: Attach detectors to all outputs and

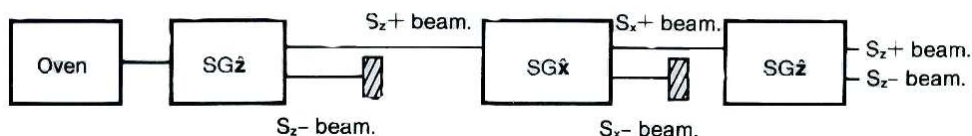


Figure 13:

explain the numbers you get using the “generalized-dot” products we have seen above.

19. Work out the above experiment for  $SG_y$ .
20. Replace  $SG_y$  with a magnetic field along an arbitrary direction (You can use the pull down menu to set angle  $\theta$  and  $\phi$  for the direction of an arbitrary magnetic field.) Plot the population on  $|SG_z^+\rangle$  and  $|SG_z^-\rangle$  as a function of  $\theta$  and  $\phi$  (Two three dimensional plots) and comment on what you see.

## 5 Homework on Pauli spin matrices and commutators:

1. Consider the *ket* vectors  $|+\rangle$  and  $|-\rangle$ . Let these *ket* vectors represent the up-spin and down-spin states of an electron along the z-orientation. (i.e.,  $|S_z^+\rangle$  and  $|S_z^-\rangle$ ) A state with spin = +1/2 and is represented by the vector  $|+\rangle$ . What is meant by this statement is that  $S_z |+\rangle = +\hbar(1/2) |+\rangle$ . The state with spin = -1/2 is represented by the vector  $|-\rangle$ . Again, this statement implies  $S_z |-\rangle = -\hbar(1/2) |-\rangle$  That is, these are *eigenstates* of the  $S_z$  operator. These two vectors form a 2-dimensional vector space that is *complete* and *orthonormal*. In matrix notation, these *ket* vectors may be written as

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5.9)$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.10)$$

This is based on the isomorphism between  $|+\rangle$  and x-polarized light and  $|-\rangle$  and y-polarized light.

Since these two vectors form a 2-dimensional vector space that is *complete* and *orthonormal* the resolution of the identity in this space can be written as:

$$\{|+\rangle\langle +| + |-\rangle\langle -|\} = I \quad (5.11)$$

2. Using these *ket* vectors the  $S_z$  operator can be represented as follows:

$$\begin{aligned} S_z &\equiv S_z \{ |+\rangle\langle +| + |-\rangle\langle -| \} \\ &= \left[ \frac{\hbar}{2} |+\rangle\langle +| - \frac{\hbar}{2} |-\rangle\langle -| \right] \\ &= \frac{\hbar}{2} [|+\rangle\langle +| - |-\rangle\langle -|] \end{aligned} \quad (5.12)$$

(Note that the quantity in square brackets [...] on the left side in Eq. (5.12) is just the identity as per Eq. (5.11). Also note that we have used  $S_z |+\rangle = +\hbar(1/2) |+\rangle$  and  $S_z |-\rangle = -\hbar(1/2) |-\rangle$  to obtain Eq. (5.12). Obtain similar expressions for  $S_x$  and  $S_y$ .

3.  $S_z$  can then be written in matrix form using the *basis-ket* vectors  $|+\rangle$  and  $|-\rangle$  as:

$$\begin{aligned} S_z &\equiv \begin{pmatrix} \langle +| S_z |+\rangle & \langle +| S_z |-\rangle \\ \langle -| S_z |+\rangle & \langle -| S_z |-\rangle \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (5.13)$$

Write down similar matrix forms for the expressions you obtained for  $S_x$  and  $S_y$  in the previous problems. These three matrices are called the Pauli-spin matrices.

4. Using the three matrices you have for  $S_x$ ,  $S_y$ , and  $S_z$ , confirm that these matrices do not commute. That is work out  $[S_x, S_y] = S_x S_y - S_y S_x$ , etc. Do these operators commute? What are the implications of this result?
5. Pauli-spin matrices are  $2 \times 2$  matrices. Which means they will act on  $2 \times 1$  vectors. As noted earlier

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5.14)$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.15)$$

And the Pauli-spin matrices can act on either these vectors or linear combinations of these vectors. Such vectors obtained from arbitrary linear combinations of  $|+\rangle$  and  $|-\rangle$  are called “spinors” (which comes from **spin**-vectors. And in general the coefficients in front of each vector  $|+\rangle$  and  $|-\rangle$  can be complex.