M Bohmian mechanics, classical mechanics as a special case of quantum mechanics, extensions to semi-classical theories such as the Wentzel Kramers Brillouin (WKB) theory:

This homework will help you see how the classical Newtonian equations arise as a special case of the time-dependent Schrödinger Equation. You will also see how alternate interpretations are possible for the wavefunction. This derivation can be a little mathematically intricate. But I would like for you to make a serious attempt to do it, since at the end of it there is some conceptual growth waiting for you.

This derivation was first performed by Madelung and de Broglie in 1926 and later by David Bohm in 1952. In fact Bohm generalized this to something called the "Bohm's interpretation of quantum mechanics" a very classical-like interpretation. In modern parlons, Bohm's interpretation is also known as the "hidden-variable" interpretation of quantum mechanics. Don't worry, I will work you through the homework:

1. Go ahead and substitute $\psi(x) = \mathcal{A}\exp\left(\imath\mathcal{S}/\hbar\right)$ in the time-dependent Schrödinger Equation. $\mathcal{A}(x,t)$ and $\mathcal{S}(x,t)$

2. Does $\psi(x) = \mathcal{A}\exp\left(\imath\mathcal{S}/\hbar\right)$ make sense? The wavefunction can be complex. And certainly any complex number can be written in this form. Do you see that? Provide a brief explanation as to why *any* complex number can be written in this form. (Hint: Say $z = x + \imath y$ is a complex number. Can you write $z = a\exp\left\{\imath\theta\right\}$? What would the values of a and θ be if this were to hold true?)

3. Having substituted $\psi(x) = \mathcal{A}\exp\left(\imath\mathcal{S}/\hbar\right)$ into the time-dependent Schrödinger Equation, go ahead and expand the terms (that is differentiate twice with respect to space, once with respect to time, etc...). After you are done doing the derivatives, you should be in a position to cancel out $\exp\left(\imath\mathcal{S}/\hbar\right)$ from both sides. (Do you understand this step of cancellation? Provide an explanation for why this can be done.)

4. Separate the real and imaginary parts to give two separate equations.

5. **Now consider the real part** equation, which should be an equation involving $\mathcal{S}$. Assume
$$\frac{1}{m}\frac{\partial\mathcal{S}}{\partial x} = v \tag{M.1}$$
where v is the velocity of x, and simplify the real part. Dont confuse this with the potential which you also have around somewhere. (Leave the $\frac{\partial\mathcal{S}}{\partial t}$ term as it is.) This substitution will make sense later.

6. Does Eq. (M.1) make sense? Look at the flux handout. See how $\mathcal{S}$ relates to the flux? What is flux interms of $\mathcal{S}$ and $\mathcal{A}$? [Remember, $\mathcal{A}^*\mathcal{A} = \rho$, the probability density. Does this make sense?]

7. How does this equation change if you were to substitute $\hbar \rightarrow 0$? Do you see that the last two terms are the classical energy of the system, that is kinetic plus potential energy?

8. The equation you have just derived after taking $\hbar \rightarrow 0$ is called the classical Hamilton-Jacobi equation. This is exactly identical to the classical Newtonian equation. How do we know? Lets see if that is true. Go ahead and substituted Eq. (M.1) in the equation you have just obtained. Simplify. Do you get F=ma? (You will have to differentiate both sides again with respect to x to get to F=ma from the Hamilton-Jacobi equation.)

9. *This exercise (so far) proves that F=ma is a special case of the time-dependent Schrödinger Equation.* Consequently, the limit $\hbar \rightarrow 0$ is called the classical limit. Do you understand this statement? Provide a brief explanation.

10. Since you have obtained the classical equations now, it is time to interpret $\mathcal{S}$. This quantity is called the *classical action*. You may (or may not) have come across this while studying classical physics. Classical action is very fundamental in classical Newtonian mechanics, which can be restated as simply "the variational minimization of classical action". (See Goldstein, Classical mechanics.)

11. However, you have (hopefully) obtained a more general expression (the real and imaginary parts) when $\hbar$ is not zero. These expressions are identical to the time-dependent Schrödinger Equation. Since you did not make any approximations to get there, just algebraic manipulations. (Clear? We just wrote the complex number $\psi(x)$ as $\mathcal{A} \exp\left(\imath \mathcal{S}/\hbar\right)$ and followed the mathematics.) However, these two expressions constitute the basics of an alternate *interpretation* of quantum mechanics called the *hidden variable theory*, and it is very fashionable now-a-days in the mathematical physics community to talk about this. :-)

12. Now what is the interpretation of the $\hbar$ dependent term that went to zero? It is called the "quantum force" or the "quantum potential". Makes sense? The Wentzel-Kramers-Brillouin (WKB) theory and further development on semi-classical theories (or quasi classical as in *partly classical partly quantum*) are based on approximations to this term. The idea in this semi-classical theory is to consider the $\hbar$ dependent quantum potential as a small perturbation term from the classical equations. A great number of people work on this area. Very rich literature starting from the time of Wigner in 1936. Sakurai gives a very nice description of how simple one-dimensional problems (like the ones we solved earlier) are solved using the WKB theory. Also see section 7.4 on the eikonal approximation.

13. My aim in the homework was to make you realize two things, (a) classical mechanics is a special case of quantum mechanics, (b) there exists is a different realm of quantum mechanics (without wavefunctions), which you see here. In fact, further half classical, half quantum approximations (or semi-classical approximations as they are called) start at this point.

14. **The imaginary part:** Make the substitution:

$$\rho(x) \equiv \psi(x)^{*}\psi(x) = \mathcal{A}^{2} \tag{M.2}$$

and simplify the equation for ρ. You will find that you obtain the continuity equation that we obtained earlier in the probability current section!! *Explain why you should have seen this coming before doing the homework.*

15. Do you see how the probability flux is related to the action?

